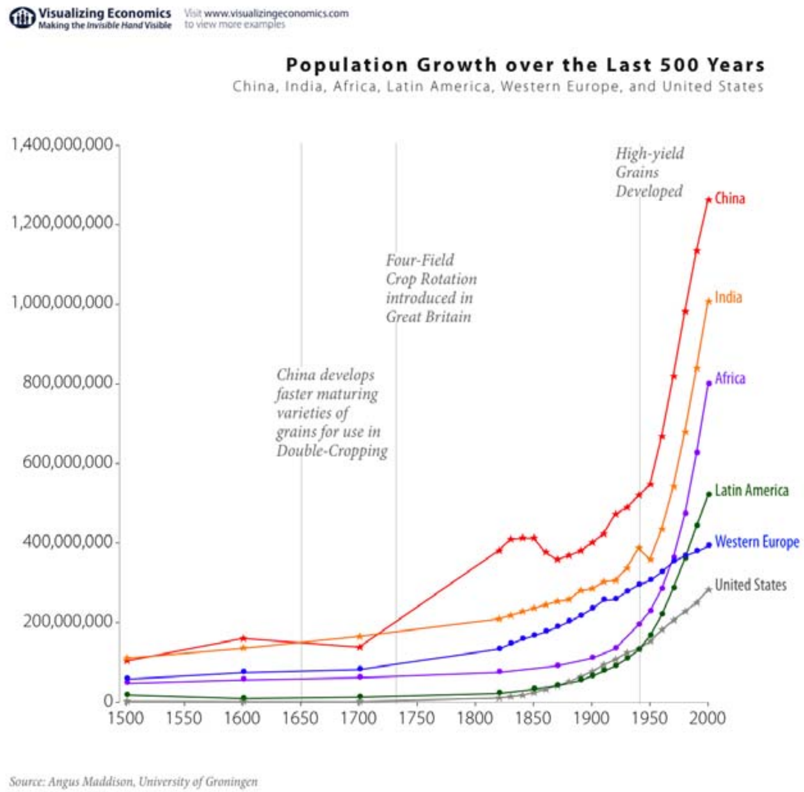
ICOM6004

The level of gross domestic product per capital of selected countries over the past 500 years.

http://visualizingeconomics.com/category/country-growth

Based on Angus Maddison

Population growth–500 years and most of the world.



http://visualecon.wpengine.netdna-cdn.com/wp-content/

Based on Angus Maddison

The issues

Plots like these are standard when making international comparisons. The idea of the course is to provide you with the tools for understanding and evaluating the reasoning based on constructions like these.

The issues covered in the course are quite diverse: they range from the question whether US per capita GDP has followed an exponential trend to whether per capita GDP is a useful measure of economic performance.

Growth mathematics

We begin with some general concepts for describing growth–e.g. of population or of income.

If the ideas are new or you want more information consult a book like

> Ian Jacques *Mathematics for Economics and Business* 6th ed. (I think the earlier editions are better!)

Notation

Let X be the quantity of interest, X_0 its value at time 0 (e.g. year 2000), X_1 its value at time 1 (e.g. 2001) and X_t its value at time t.

Linear growth ("simple interest")

Here X changes by a constant amount, c, each period. In finance this is called **simple interest**. If $c > 0$ there is growth if $c < 0$ decay.

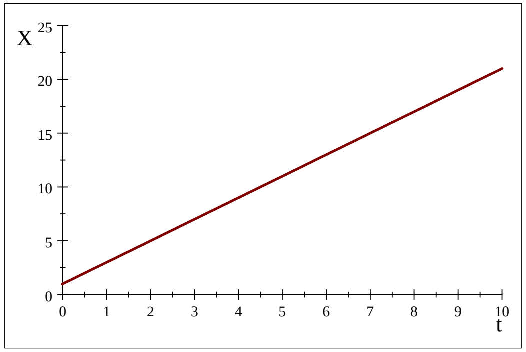
Consider the first two values

$$X_1 = X_0 + c$$
$$X_2 = X_1 + c$$
$$\Rightarrow X_2 = (X_0 + c) + c = X_0 + 2c.$$

Generalising

$$X_t = X_{t-1} + c \text{ for } t = 1, 2, \ldots \text{ and } X_0 \text{ fixed}$$
$$\Rightarrow X_t = X_0 + tc.$$

This equation relating X to t is a **linear** equation. Plotting X against t for the case $c = 2$ and $X_0 = 1$ gives



Linear trend $X = 1 + 2t$

In 1 period X increases by 2, in 5 periods X increases by 10, in 10 by 20, ..., in 50 by 50.

The growth rate defined as $\frac{X_t - X_{t-1}}{X_{t-1}}$ declines over time

$$X_t = X_{t-1} + c \text{ and } X_{t-1} = X_0 + (t-1)c$$
$$\Rightarrow \frac{X_t - X_{t-1}}{X_{t-1}} = \frac{c}{X_0 + (t-1)c}.$$

As t increases, this ratio diminishes. If $c = 0.2$ and $X_0 = 1$, the growth rate at $t = 1$ is 0.2 but for $t = 10$ it is $0.2/(1 + 1.8) = 0.07$.

Geometric growth (“**compound interest**”)

Instead of the change, $X_t - X_{t-1}$, being constant each period the growth rate might be constant. This is so for geometric growth where the ‘dynamic’ is

$$X_1 = cX_0$$
$$X_2 = cX_1$$
$$\Rightarrow X_2 = c(cX_0) = c^2X_0.$$

Generalising

$$X_t = cX_{t-1} \text{ for } t = 1, 2, \ldots \text{ and } X_0 \text{ fixed}$$
$$\Rightarrow X_t = c^tX_0$$

The sequence of values $X_0, cX_0, c^2X_0, \ldots$

is called a **geometric sequence**–giving this kind of growth its name.

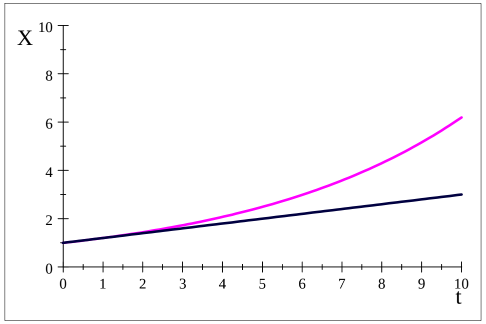
The growth rate is constant over time because

$$\begin{aligned} X_t &= cX_{t-1} \\ \Rightarrow \frac{X_t - X_{t-1}}{X_{t-1}} &= \frac{cX_{t-1} - X_{t-1}}{X_{t-1}} \\ &= c - 1. \end{aligned}$$

Thus if $c = 1.2$ the growth rate is 0.2 or 20%. If $0 < c < 1$, the system decays rather than grows.

Linear and geometric trends

If the change $c = 1.2$ and the starting value $X_0 = 1$ then a plot of X against t looks like



Linear and geometric trends

X is 20% larger after 1 period, around 2.5 times larger after 5 periods, around 6 times larger after 10 and nearly 40 times larger after 20.

Some historical figures (from Barro and Sala-Martin *Economic Growth* p. 1):

Real per capita GDP in the US grew by a factor of 8.1 from 1870 to 1990. This corresponds to a growth rate of 1.75% per annum.
If the growth rate had been 0.75% (a little more than that of India 1900-87) the expansion factor would have been 2.5.

Doubling time and the Rule of Seventy (P&B 274)

If X is growing geometrically, how long does is it take to double in size?

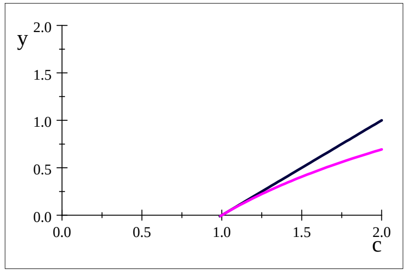
Given that $X_t = c^t X_0$ doubling requires $c^t = 2$.

To find t, take (natural) logarithms and solve

$$t \ln c = \ln 2 \Rightarrow t = \frac{\ln 2}{\ln c}.$$

Now $\ln 2 = 0.693$, approx. 70%. The population of Niger is growing at 3.3% per annum: so $c = 1.033$, $\ln c = 0.032$ and the doubling time, $t = 21$ years.

The **demographers**’ **rule of seventy** is based on noting that $\ln c \approx c - 1$ for c near 1 and so a good approximation to t can be obtained by dividing 70 by the percentage growth rate.



$\ln c$ and $c - 1$

For Niger the doubling time is roughly $70/3.3 = 21$ years. By contrast the annual growth rate for the Netherlands is 0.3% with doubling time roughly $70/0.3 = 233$ years.

Discrete and continuous time

I have treated time as a discrete quantity

$$t = 0, 1, 2, \ldots$$

corresponding to years or months–January, 2000, February, 2000 etc.

BUT in my diagrams time is treated as a continuous variable with the values between the integers, $0, 1, 2, \ldots$, filled in. We don’t jump from January to February but pass through every instant of January.

In theoretical work time is often treated as a continuous variable. The main advantage of ‘working in continuous time’ is that calculus can be used. See Jacques ch. 4

The form of growth called geometric growth in discrete time is called **exponential growth** in continuous time. The relation between X and t is given by the exponential function

$$X(t) = ae^{bt} \text{ for } t \geq 0.$$

where a and b are constants; see

Jacques 182ff.

Consider X at time $t-1$. We have

$$X(t-1) = ae^{b(t-1)} = e^{-b}ae^{bt}$$
$$\Rightarrow X(t) = e^{b}X(t-1).$$

So e^b corresponds to c in the discrete treatment.

Using calculus we can get an interpretation for b:

$$X(t) = ae^{bt}$$
$$\Rightarrow \begin{cases} \frac{dX}{dt} = bae^{bt} \\ \frac{dX}{dt} \bullet \frac{1}{X} = b. \end{cases}$$

The quantity $\frac{dX}{dt}\frac{1}{X}$ is called the **instantaneous rate of growth**. For an exponential function the rate of growth is constant.

We can also write

$$\frac{dX}{dt} = bX$$

and say that ae^{bt} is a solution of this **differential equation** as c^tX_0 is a solution of the **difference equation**

$$X_t = cX_{t-1}.$$

Difference and differential equations can be pursued in Jacques Additional Topic 2.

The behaviour of ratios

Suppose income, $X(t)$, and population, $N(t)$, are both growing exponentially.

$$X(t) = ae^{bt} \text{ and } N(t) = ce^{dt}$$
$$\Rightarrow \frac{X(t)}{N(t)} = \frac{ae^{bt}}{ce^{dt}} = \frac{a}{c} \bullet e^{(b-d)t}.$$

The instantaneous rate of growth of income per head is $b-d$. Jacques has more on exponential functions.

Fitting trend curves

Often we have a time series, say a sequence of population figures like those for the US represented in the second figure above, and we wish to find the best fitting trend line to the points.

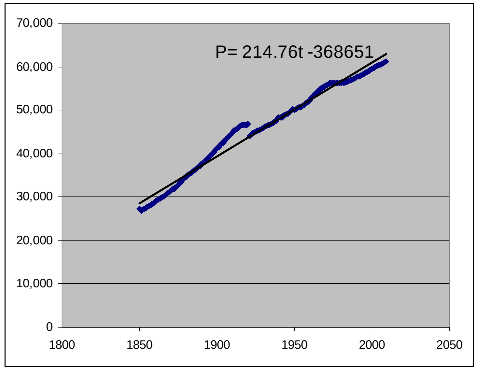
The points seldom lie exactly on any simple trend curve and it is usual to imagine that X_t is equal to the trend value with a random 'error'

$$X_t = \text{ trend value} + \text{random term.}$$

The trend may be linear, exponential or some other simple function of time.

The line of best fit is the line which is closest to the observed values in the sense that the sum of squared vertical deviations of point from line is a minimum. The method is called **least squares**.

Here are some results I obtained (using Excel) for the population of the United Kingdom 1850-2009.



I have fitted a straight line with slope 2175.

In the exercises I ask you to do similar calculations.