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AN ALTERNATIVE DERIVATION OF DURBIN'S h STATISTIC

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DURBIN [**1**, p. 411] HAS DEVELOPED a test of H_0 when "we have a family of models giving, for a sample of size n, likelihood $L(\alpha, \beta)$ depending on two vectors α and β of p and q parameters. β denotes the set of parameters of primary interest and α denotes a set of nuisance parameters. The null hypothesis is $H_0: \alpha = \alpha_0$ and we suppose that we are in a situation where the analysis is much easier when H_0 is true than when it is false." Durbin develops a test procedure and establishes its asymptotic equivalence to the likelihood ratio test. However this is exactly the situation envisaged for application of the efficient score test developed in Rao [**2**] (and discussed in Silvey [**3**] under the name χ^2 test). This efficient score test or Lagrangian multiplier test is distinct from Durbin's test although asymptotically equivalent to it.

The aim of this note is to show that the h test for serial correlation in least squares regression when some of the regressors are lagged dependent variables can also be derived from the efficient score principle. The notation and assumptions follow Durbin:

$$y_t = \beta_1 y_{t-1} + \ldots + \beta_{r+1} x_{1t} + \ldots + \beta_{r+s} x_{st} + u_t \qquad (t = 0, \ldots, n),$$

where $u_0, \ldots, u_n$ are a sample from

$$u_t = \alpha u_{t-1} + \xi_t, \quad |\alpha| < 1 \qquad (t = 1, \ldots, n)$$

Assume: (i) (ξ_t) is a sequence of independent $N(0, \beta_{r+s+1})$ variables; (ii) $y_0, y_{-1}, \ldots, y_{-r}$ and $x_{10}, \ldots, x_{s0}$ are known constants; u_0 is constant but unknown; (iii) the x's are exogenous and $(1/n)\Sigma_{t=1}^{n} x_{it} x_{jt}$ $(i, j = 1, \ldots, s)$ converges to a finite positive definite limit as $n \to \infty$; (iv) the roots of $x^r = \Sigma_{i=1}^{r} \beta_i x^{r-i} = 0$ all have modulus less than one.

To test the null hypothesis $H_0: \alpha = 0$ we follow the procedure of Silvey [**3**, Section 7.4.1] and form the statistic

$$\begin{pmatrix} \dfrac{\partial \log L(0, b)}{\partial \alpha} \\ \dfrac{\partial \log L(0, b)}{\partial \beta} \end{pmatrix}' (n\mathscr{I})^{-1} \begin{pmatrix} \dfrac{\partial \log L(0, b)}{\partial \alpha} \\ \dfrac{\partial \log L(0, b)}{\partial \beta} \end{pmatrix} \sim \chi_1^2$$

where the partial derivatives are evaluated at the values of the constrained ML estimates of α and β, respectively O and b. $\mathscr{I}$ is the estimated information matrix. Now

$$\begin{aligned} \frac{\partial \log L(0, b)}{\partial \alpha} &= \frac{1}{b_{r+s+1}} \Sigma(y_t - b_1 y_{t-1} - \ldots - b_r y_{t-r} - b_{r+1} x_{1t} - \ldots \\ &\quad - b_{r+s} x_{st})(y_{t-1} - b_1 y_{t-2} - \ldots - b_r y_{t-r-1} \\ &\quad - b_{r+1} x_{1t-1} - \ldots - b_{r+s} x_{st-1}) \\ &= \frac{1}{b_{r+s+1}} \sum z_t z_{t-1} \end{aligned}$$

where the z's are the least squares residuals, but $\partial \log L(0, b)/\partial \beta = 0$. $\mathscr{I}$ is derived in [**1**]

[1] I am grateful to the referees for their helpful comments.

and is

$$\left[\begin{array}{c|c|c} 1 & 1 \quad 0 \quad \dots \quad 0 & 0 \\ \hline \begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} & \operatorname{plim} \dfrac{1}{n\beta_{r+s+1}} X'X & 0 \\ \hline 0 & 0 & \dfrac{1}{2\beta_{r+s+1}^2} \end{array}\right].$$

with β_{r+s+1} replaced by b_{r+s+1}.

The test statistic is therefore

$$\left(\frac{\sum z_t z_{t-1}}{b_{r+s+1}}\right)^2 \frac{n}{1-n\hat{v}(b_1)} = \hat{K}_2$$

as given in expression (11) of [**1**] where $\hat{v}(b_1)$ is the estimate of the variance of b_1. A rearrangement gives the h statistic.

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REFERENCES

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[**2**] RAO, C. R.: "Large Sample Tests of Statistical Hypotheses Concerning Serial Parameters with Applications to Problems of Estimation," *Proceedings of the Cambridge Philosophical Society*, 44 (1948), 50–57.

[**3**] SILVEY, S. D.: *Statistical Inference*. Middlesex, England: Penguin Books Ltd., 1970.